

P.Grinevich, S.Novikov ¹**Discrete SL_2 Connections and Self-Adjoint Difference Operators on the Triangulated 2-manifolds**

Abstract. *Discretization Program of the famous Completely Integrable Systems and associated Linear Operators was developed in 1990s. In particular, specific properties of the second order difference operators on the triangulated manifolds were studied in the works of S.Novikov and I.Dynnikov since 1996. They involve factorization of operators, the so-called Laplace Transformations, New Discretization of Complex Analysis and New Discretization of GL_n Connections on the triangulated n -manifolds. The general theory of the new type discrete GL_n connections was developed. However, the special case of SL_n -connections was not selected properly. Indeed, it appears in the theory of important self-adjoint operators. In the present work we construct a Theory of SL_2 discrete connections on the triangulated 2-manifolds. They are deeply associated with real self-adjoint difference operators similar to complex line bundles (magnetic fields) in the 2nd order Schrodinger operators on the plane in the standard continuous quantum mechanics.²*

1. Discrete GL_n Connections.

Let M^n be a triangulated manifold equipped by the set of (real positive) numbers $u_{T:P} \neq 0$ assigned to every n -simplex T and its vertex $P \in T$. They define an operator Q mapping functions of vertices to the functions of n -simplices

$$Q\psi_T = \sum_P u_{T:P} \psi_P$$

Definition 1. *The Discrete GL_n Connection is defined by the equation $Q\psi = 0$ up to the arbitrary Abelian Gauge Transformations defined by the pair of nonzero functions of triangles and vertices $g_T > 0, h_P > 0$:*

$$\psi \rightarrow h_P \psi_P, \mu_{PP'}^T \rightarrow h_{P'} / h_P \mu_{PP'}^T, Q \rightarrow g_T Q / h_P$$

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²The works of S.Novikov with and without collaborators quoted here can be found in his homepage www.mi.ras.ru/~snovikov click Publications, items 136,137,138,140, 146,148,159,163, 173,174,175.

It means in particular that only ratios

$$\mu_{PP'}^T = u_{T:P}/u_{T:P'} > 0$$

are important for the connection $Q\psi = 0$. Take any "framed path" $\gamma^{fr} = [P_0P_1\dots P_m; T_1, T_2, \dots, T_m]$ where $P_{i-1}P_i$ is an edge in the n -simplex $T_i, i = 1, \dots, m$. Every such framed path defines an **Abelian Framed Holonomy Representation** assigning to the framed path a number

$$\gamma^{fr} \rightarrow \prod_i \mu_{P_{i-1}P_i}^{T_i} = \mu(\gamma^{fr})$$

The semigroup $\Omega_{framed}(M, P_0)$ of the closed framed paths with initial vertex P_0 maps into the multiplicative group k^* of our basic field which we take here as R^+ only:

$$\mu : \Omega_{framed} \rightarrow k^*$$

The initial point is unimportant because everything is abelian here. **The expressions $\mu(\gamma^{fr})$ are gauge invariant.** The simplest of them are

$$\rho_{PP'}^{TT'} = \mu_{PP'}^T \mu_{P'P}^{T'} = \rho_{P'P}^{T'T} = \mu_{PP'}^T / \mu_{PP'}^{T'}$$

corresponding to the framed paths $\gamma^{fr} = [PP'P; TT']$.

The Reconstruction of Connection for $n = 2$. In order to reconstruct connection we need to know all $\rho_{PP'}^{TT'}$ and also collection of $\mu(\gamma_j^{fr})$ representing the set of generators of the group $H_1(M, Z)$. Corresponding procedure is presented in the work [9].

Let M be an oriented 2-manifold.

We define quantity

$$\rho(T) = \prod_{R_q \in T} \rho_{R_q}^{TT_q}$$

where $T_q \cap T = R_q = ij, T = ijk, R_q$ is an edge in T . This multiplicative 2-cochain is homologous to zero (see proof in [9]). In particular, for the compact closed orientable manifold we proved that

$$\prod_{T \in M} \rho(T) = 1$$

Let $\delta\lambda = \rho^{-1/2} > 0$, so we have

$$\lambda_{ij}\lambda_{jk}\lambda_{ki} = \rho^{-1/2}(T), T = ijk, \lambda_{ij}\lambda_{ji} = 1$$

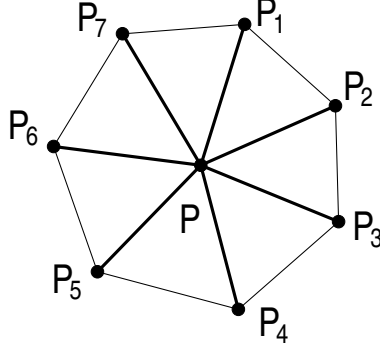


Figure 1:

We define finally

$$\mu_{ij}^T = \lambda_{ij} / (\rho_{ij}^{TT_q})^{1/2}$$

This formula solves our problem following the work [9].

The cochain λ is nonunique: one can multiply it by the arbitrary co-cycle. In fact, they can be changed in the cohomology classes because the abelian gauge transformations of the connection acts exactly in that way. It lead to the invariants of connections in the homology group $H_1(M)$ additional to the quantities like ρ .

A Nonabelian Holonomy Representation is defined by the closed "Thick Paths" $\gamma^{thick} = T_1 \dots T_m$ where $F_i = T_i \cap T_{i+1}$ is a common $n - 1$ -face, $F_i \neq F_{i+1}$:

$$K : \Omega_{thick}(M, T_0) \rightarrow GL_n(R)$$

A Nonabelian Curvature in the point P for $n = 2$ is defined as a Nonabelian Holonomy corresponding to the Thick Path $\gamma_P^{thick} = T_1 \dots T_m$ consisting of 2-simplices $T_i = PP_iP_{i+1}$ surrounding vertex P —see Fig 1

The star of this vertex is $St(P) = T_1 \cup T_2 \cup \dots \cup T_m$. Nonabelian Curvature is an upper triangle 2×2 -matrix $K(P, T_1)$ with diagonal $(1, \mu_P)$ where $\mu_P = \mu(\gamma_P^{fr})$ and second row $(\alpha(P_i, P), \mu_P)$. Here γ_P^{thick} is a closed thick path ∂St_P starting and ending in any vertex $P_i \in T_i \in St_P, P_i \neq P$. Easy to see that following lemma is true:

Lemma 1. *For the closed paths surrounding vertex P we have*

$$\mu_P = \prod_{i=1, \dots, m} \rho_{PP_i}^{T_i T_{i+1}}$$

where index i is varying modulo m .

In the work [9] one can find all expressions for the coefficients $\alpha(P_i, P)$ of the Nonabelian Curvature through the gauge invariant quantities $\mu_{PP'}^T \rho_{PP'}^{TT'}$.

Problem: How to construct effectively all discrete SL_2 connections on the triangulated manifold M^2 ?

Construction. We use the following procedure. Assign to every edge $R \in M$ a number $A(R) > 0$. For every triangle $T \in M$ we have exactly 3 vertices P_1, P_2, P_3 and 3 edges R_1^T, R_2^T, R_3^T . Here $R_i = P_i P_{i+1}$, and i is counted modulo 3. We define the Connection Operator Q by the equations

$$Q\psi_T = \sum_i u_{T:P_i} \psi_{P_i}$$

$$u_{T:P_i} u_{T:P_{i+1}} = A(P_i P_{i+1})$$

Theorem 1. *Discrete Connection defined by such operator Q is locally and globally SL_2 . Every SL_2 connection in the simply connected manifold can be obtained by that construction.*

Proof. For every edge $R = PP'$ and 2 triangles $T = PP'S, T' = P'PS'$ where $T \cap T' = R$ we defined above a gauge invariant function

$$\rho_{PP'}^{TT'} = u_{T:P}^{-1} u_{T:P'} u_{T':P'}^{-1} u_{T':P}$$

Taking into account relations between the products of the connection coefficients, we are coming to the following result

$$\rho_{PP'}^{TT'} = u_{T':P}^2 / u_{T:P}^2 = u_{T:P'}^2 / u_{T':P'}^2$$

In particular it means that functions ρ for the edges PP' can be calculated from data in one vertex only. Consider the series of triangles T_1, T_2, \dots, T_k with the same vertex $P \in T_i = PP_i P_{i+1}$ forming a thick path $T_i \cap T_{i+1} = PP_i$. We have

$$\prod_{i=l}^{i=k} \rho_{PP_{i-1}}^{T_{i-1} T_i} = \mu(\gamma^{fr})$$

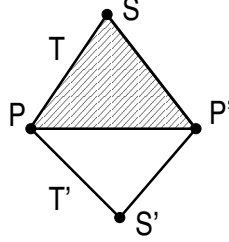


Figure 2:

where

$$\gamma^{fr} = [PP_l \dots P_{k-1} P_k P; T_{l-1} T_{l-1} T_l \dots T_{k-1} T_k T_k]$$

For the whole cycle $\partial St(P)$ around the vertex P this equality implies $\mu(\gamma^{fr}) = 1$, because $l = 1, P_1 = P_k$ in this case. The edge PP_1 disappears from the path. So our connection is locally SL_2 .

In order to prove that the connection is globally SL_2 , we consider any thick path $\gamma^{thick} = T_0 T_1 \dots T_m$ starting from the edge $R_0 = P_0 P'_0$. Following the work [9], every thick path is defined by the word in the free associative semigroup with 2 generators a_1, a_2 : either this word starts from a_1 and looks like $a_1^{p_1} a_2^{q_1} \dots$ or it starts from a_2 , i.e. we have $a_2^{q_1} a_1^{p_1} \dots$ where q_j, p_j are positive integers, $j = 1, 2, \dots, k$. We present this picture as a pair of 2 vertical boundary half-lines starting from the initial vertices P_0 and P'_0 of the form $P_0 P_1 P_2 \dots$ and $P'_0 P'_1 P'_2 \dots$. This half-lines l, l' with the initial edge form the boundary of the "diagram" defining thick path (i.e. the triangulated strip in the half-plane above the edge $P_0 P'_0$) where all edges are either "vertical"–i.e. belong to the half-lines l, l' or "horizontal"–i.e. join l and l' –see Fig 3.

We are going to apply the "product equality" for the horizontal ρ :

$$\prod_{R_j} \rho_{R_j}^{T_j - 1 T_j} = \mu(\gamma^{fr})$$

where R_j are all horizontal edges in the closed thick path and $\mu(\gamma^{fr}) = \mu(l)\mu(l')$. Here l', l are the boundary framed closed paths for the thick path with framing given by the triangles T_j in the strip.

As we see, at first we are going around the first center $P_0 \in l$ exactly p_1 times, after that we shift center to the the vertex P'_{p_1} at the line l' and

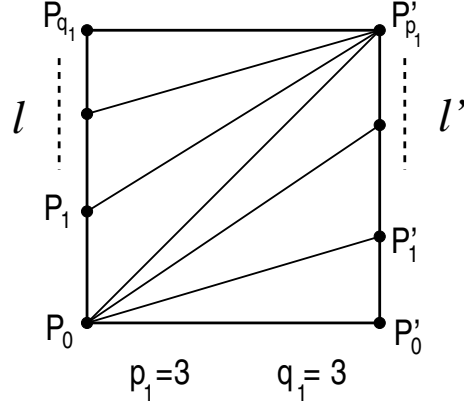


Figure 3:

rotate around it q_1 times, and so on. For each such step we use the previous product formula. The quantities $\rho_{PP'}^{TT'}$ in this case can be expressed as a ratios of the squares of coefficients of the connection operator Q attached to the same vertex P or P' from the right and left sides of this edge. So we can jump from the line l to the line l' and back in process of calculation, when corresponding rotation ends. Finally, for the closed thick path we are coming to the same initial coefficients but in the opposite powers like in the case of curvature. So the product is equal to one.

This argument concludes the proof that this construction really leads to some SL_2 connection. It depends on the number of parameters equal to the number R of edges in the case of compact manifolds. The number of gauge classes is equal to the difference $R - V$ neglecting some finite topological constants independent on triangulation. Here V is a number of vertices.

Let some locally SL_2 connection be given as a collection of connection coefficients $\mu_{PP'}^T > 0$. We choose an arbitrary edge $R = PP'$ and number $A(PP') > 0$. After that we can find all coefficients of the Operator Q for this Connection starting from triangles T, T' containing R . We extend this construction to the neighboring triangles. Obviously, we meet no obstruction for simply connected manifolds. The only obstructions might appear extending this construction along the thick paths nonhomologous to zero. So we will get the determinant of the nonabelian holonomy map for the semigroup of thick paths reducing the abelian part $\pi_1(M) \rightarrow k^* / \pm 1 = R^+$. So our theorem is proved for simply connected manifolds.

In general we construct in such way all SL_2 connections in simply connected manifolds only. Otherwise, the process of extension of the edge functions $A(R)$ leads to nontrivial holonomy: the Bloch multipliers appear. Periodicity of the connection coefficients μ does not imply periodicity of the edge function $A(R)$ obtained by the process of extension above.

So we are coming to all locally SL_2 connections on M making our extension at the abelian coverings $\hat{M} \rightarrow M$ over the manifold M .

2. SL_2 -Connections and Difference Self-Adjoint Operators.

Consider here any triangulated 2-manifold M^2 with "discrete conformal structure" which is defined as a coloring of 2-simplices (triangles) into the black and white colors, according to the works [8, 10, 12, 13]. As we observed in [1, 2, 4], every scalar real self-adjoint second order operator L can be presented here in the factorized form ("black" and "white"):

$$L\psi_P = \sum_{P'} b_{P:P'} \psi_{P'} = \sum_{P \neq P'} b_{P:P'} \psi_{P'} + W(P) \psi_P$$

where either $P = P'$ or PP' is an edge. Let $b_{P:P'} = b_{P':P}$. We call the set of coefficients $b_{P:P}$ "Potential" $W(P) = b_{P:P}$ and require $b_{P:P'} > 0$ for all PP' . There exist unique operators Q^b and Q^w —the black and white triangle operators—such that

$$L = Q^{b+} Q^b + W^b = Q^{w+} Q^w + W^w$$

Here $Q^b \psi_T = \sum_{P \in T} u_{T:P} \psi_P$ where T is any black triangle, and $Q_T^w = \sum_{P \in T} u_{T:P} \psi_P$ where T is any white triangle (see Fig 4).

Consider now the the combined operator $Q_L = \{Q^b, Q^w\}$. It defines some discrete GL_2 Connection in the manifold M^2 associated with self adjoint real operator L .

Theorem 2. *The Connection Q_L is an SL_2 Connection.*

Proof. Every edge $R = PP'$ is intersection of black and white triangles $R = T \cap T'$. From the equality $Q^{b+} Q^b = Q^{w+} Q^w$ modulo diagonal part we conclude that every nondiagonal coefficient of operator L is presented as a product twice through the coefficients of operators Q^b and Q^w correspondingly:

$$u_{T:P} u_{T':P'} = u_{T':P} u_{T:P'} = A(PP')$$

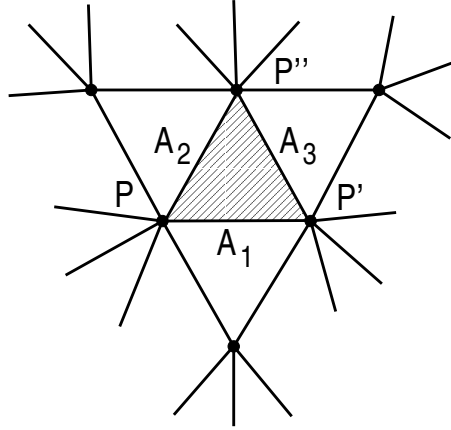


Figure 4:

However, it was proved above that the set of numbers $A(R)$ for every edge $R = PP'$ defines a SL_2 Connection. Theorem is proved.

3. Factorizations, Laplace Transformations and Discrete Analog of 2D Toda Lattice.

Various factorization of linear operators acting on the scalar functions in R^D leads to the so-called **Darboux Transformations** where $L\psi = \lambda\psi$:

$$L = Q^+Q \rightarrow QQ^+ = \tilde{L}, \psi \rightarrow \tilde{\psi} = Q\psi$$

and **Laplace Transformations**—see [1, 2, 3, 4, 6, 9, 7, 11, 12].

Here $L\psi = 0$

$$L = Q^+Q + W \rightarrow W^{1/2}QW^{-1}Q^+W^{1/2} + W = \tilde{L}$$

$$\psi \rightarrow \tilde{\psi} = W^{-1/2}Q\psi$$

In order to apply Laplace transformation, the product $W^{-1}Q\psi$ should be well-defined. However, the operator Q maps in several cases above the initial space of functions into another functional space. At the same time the potential W maps the initial space into itself, so our product is not well defined in general. The Darboux transformation always can be applied but we need property $W = 0$ for our factorization which is not true in many cases. For $D = 2$ we found following classes only where Laplace transformation is always well-defined and can be nontrivially iterated infinite number of times.

I. The Square Lattice. The operators L acting in the space of functions of vertices in the square lattice with basic shifts T_1, T_2 , of the "hyperbolic form" (see [1, 2])

$$L = a(m, n) + b(m + 1, n)T_1 + c(m, n + 1)T_2 + d(m + 1, n + 1)T_1T_2$$

with gauge equivalence group of the "hyperbolic equation" $L\psi = 0$:

$$L \rightarrow fLg, \psi \rightarrow g^{-1}\psi$$

where both functions f, g are everywhere nonzero. We have 2 factorizations ("right" and "left")

$$L = f[(1 + uT_1)(1 + vT_2) + w] = g[(1 + vT_2)(1 + uT_1) + w_L]$$

Laplace transformation is defined as usual

$$\tilde{L} = f'[(1 + vT_2)w^{-1}(1 + uT_1) + 1]$$

In order to apply Laplace Transformation, we use representative L in the zero level gauge class where $L = (1 + uT_1)(1 + vT_2) + w$. Here $u = u(m + 1, n)$, $v = v(m, n + 1)$, $w = w(m, n)$. For the Laplace Image \tilde{L} corresponding coefficients are u', v', w' . They are taken in the same points.

We have to choose f' properly to return to the similar gauge form after transformation $f' = (1 + w')w/(1 + w)$:

$$\tilde{L} = f'[(1 + vT_2)w^{-1}(1 + uT_1) + 1] = (1 + u'T_1)(1 + v'T_2) + w'$$

Here we use notations for the shifts of functions $u_i = T_i^*(u)$, $v'_i = T_i^*(v)$,

After substitution, we have:

$$u' = f'u/w, v' = f'v/w_2, u'_1 v'_1 = f'vu_2/w_2$$

We still have gauge transformation preserving this form $L \rightarrow f^{-1}Lf, \psi \rightarrow f\psi$. The invariants of the last gauge transformations are following: The potential w in the factorized form and **Curvature** $H = vu_2u^{-1}v_1^{-1}$ similar to magnetic field in the continuous case. After Laplace Transformation we get result

$$H' = (1 + w'_2)/(1 + w_2), 1 + w' = (1 + w)w_{m-1,n}w_{m,n+1}w^{-1}w_{m-1,n+1}^{-1}H_{m-1,n}$$

or after shift T_1 :

$$1 + w'_1 = (1 + w_1)ww_1^{-1}w_{1,2}^{-1}H$$

So, following the work [2]³, we can express the invariants H', w' through the invariants H, w only similar to the continuous case. We exclude H, H' expressing all Chain through the variable w (see [4]). We are coming finally to the discretization of the 2D Toda Lattice following the classical scheme of Darboux and his school in the XIX Century and based on the Laplace Chains (see quotations in the work [4]). Similtaneously it was done also in [5]. So the Laplace Chain

$$\dots \rightarrow L_{k-1} \rightarrow L_k \rightarrow L_{k+1} \rightarrow \dots$$

is described by the equation

$$\frac{w''_1 + 1}{w'_1 + 1} \times \frac{w_2 + 1}{w'_2 + 1} = \frac{w'w'_{1,2}}{w'_1w'_2}$$

³There were mistakes in the formulas for the equation in the work [2] corrected here.

where f' means the same function for the result of Laplace Transformation, f'' means the same made twice. In particular w, w', w'' correspond to the numbers $k-1, k, k+1$ in our chain $w^k(m, n) = w', w^{k-1}(m, n) = w, w^{k+1}(m, n) = w''$ in the point (m, n) , w_i means shift of this function to the direction $T_i, i = 1, 2$, i.e. $w_{m+1, n}$ and $w_{m, n+1}$ correspondingly: so the equation for the quantity $w^k(m, n)$ is following

$$\frac{w_1^{k+1} + 1}{w_1^k + 1} \times \frac{w_2^{k-1} + 1}{w_2^k + 1} = \frac{w^k w_{12}^k}{w_1^k w_2^k}$$

Example. Following [2], consider a Cyclic Chain Problem here for the case of period equal to 2 leading in the continuous case to the sin(sinh)-gordon equations. Let $a = w^{2k}, b = w^{2k+1}$ for all k . We get 2 equations for $a(m, n)$ and $b(m, n)$. They imply that for the quantity $G = ab$ we have

$$\frac{GG_{12}}{G_1 G_2} = 1$$

Consider now only its partial solution $G = C$ where C is a constant. Finally we are coming to the system

$$\frac{C + b_1}{1 + b_1} \times \frac{C + b_2}{1 + b_2} = bb_{12}$$

It can be viewed as some discrete analog of the sinh-gordon system as it was pointed out already in [2] in 1997. For $C = 1$ this system degenerates to the trivial form $bb_{12} = 1$.

The equivalence of this discretization of the 2D Toda Lattice with known systems should be checked: its connection with totally discrete 3D Hirota system was informally claimed by one of the authors of this work many years ago but actually never was checked properly. Many people studied the square lattice at that time in 1990s. It should be checked whether this question was clarified in the literature or not.

The family of Hirota systems can be written in the form

$$\begin{aligned} \gamma F(k+1, m+1, n)F(k-1, m, n+1) + \alpha F(k, m, n)F(k, m+1, n+1) + \\ + \beta F(k, m+1, n)F(k, m, n+1) = 0 \end{aligned}$$

assuming that $\alpha + \beta + \gamma = 0$.

It can be written in the form similar to our system if $\gamma \neq 0$:

$$\frac{v_1^{k+1}v_2^{k-1}}{v_1^k v_2^k} = \lambda + (1 - \lambda) \frac{v^k v_{1,2}^k}{v_1^k v_2^k}$$

where $F(k, m, n) = v^k(m, n)$ and $v_i^k = T_i^*(v^k)$ as above are shifts along the directions m, n in the selected plane.

Our discretization of the 2D Toda system also can be written in the similar form

$$\frac{v_1^{k+1}v_2^{k-1}}{v_1^k v_2^k} = \frac{(v^k - 1)(v_{1,2}^k - 1)}{(v_1^k - 1)(v_2^k - 1)}$$

Here $v^k(m, n) = w^k(m, n) + 1$ in the previous notations. After scaling transformation $v \rightarrow \kappa v$ we can rewrite this system in the form

$$\frac{v_1^{k+1}v_2^{k-1}}{v_1^k v_2^k} = \frac{(v^k - \kappa)(v_{1,2}^k - \kappa)}{(v_1^k - \kappa)(v_2^k - \kappa)}$$

with $\kappa \neq 0$. In the limit $\kappa \rightarrow 0$ we are coming to the partial degenerate case of Hirota systems otherwise these systems are different. At the same time, they are written through the same geometric invariants associated with the square lattice.

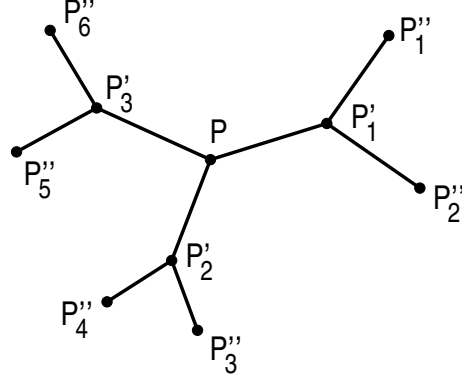


Figure 5:

II. The Trivalent Tree. The real selfadjoint 4th order operators L acting on the functions of vertices in the trivalent tree Γ —see[7]. We have here

$$L\psi_P = \sum_{P''} a_{PP''}\psi_{P''} + \sum_{P'} b_{PP'}\psi_{P'} + W_P\psi_P$$

where $PP'P''$ is a short path of the length 2, $|PP'| = 1$ and $|P'P''| = 1$ —see Fig 5

It always can be factorized in the form $L = Q^+Q + u_P$ where

$$Q\psi_P = \sum_{P'} d_{PP'}\psi_{P'} + v_P\psi_P$$

$$a_{PP''} = d_{P'P}d_{P'P''}, b_{PP'} = d_{P'P}v_{P'} + d_{PP'}v_P$$

$$W_P = v_P^2 + \sum_{P'} d_{P'P}^2 + u_P$$

Laplace transformation is defined as usual

$$\tilde{L} = Qu^{-1}Q^+ + 1, \tilde{\psi} = Q\psi$$

in the self-adjoint zero level equivalence class $L \rightarrow fLf, \psi \rightarrow f^{-1}\psi$. The iteration of Laplace transformation leads to the Laplace Chains

$$\dots \rightarrow L_n \rightarrow L_{n+1} \rightarrow \dots$$

where $\tilde{L}_n = L_{n+1}$. In this case the choice of factorizations depends on parameter because the equation

$$b_{PP'} = d_{P'P}v_P + d_{PP'}v_{P'}$$

has an one-parametric family of solutions for v depending on the initial value v_{P_0} in the selected point $P_0 \in \Gamma$. We perform Laplace transformations every time choosing different parameters for the solution of this equation. Therefore Laplace Chains may be nontrivial and long. They are similar to the Darboux Chains in the 1D case of the continuous Sturm-Liouville Operators studied by J.Weiss, A.Shabat and A.Veselov [14, 15, 16], where factorization also depends on parameter numerating solutions of the Riccati Equation.

III. The Equilateral Triangle Lattice. The real self-adjoint second order operators L acting in the space of functions of vertices in the equilateral triangle lattice with shifts $T_1^{\pm 1}, T_2^{\pm 1}, (T_1^{-1}T_2)^{\pm 1}$ of equal length

$$L = a(m, n) + \{b(m+1, n)T_1 + c(m, n+1)T_2 + \\ + d(m-1, n+1)T_1^{-1}T_2 + (adjoint)\}$$

where $T_i^+ = T_i^{-1}$ We have here right and left factorizations

$$L = Q^+Q + W, Q = (u + vT_1 + wT_2)$$

and

$$L = Q'^+Q' + W', Q' = u' + v'T_1^{-1} + w'T_2^{-1}$$

This lattice defines naturally a black-white colored triangulation of the plane R^2 where $Q = Q^b$ and $Q' = Q^w$ —see Fig 6.

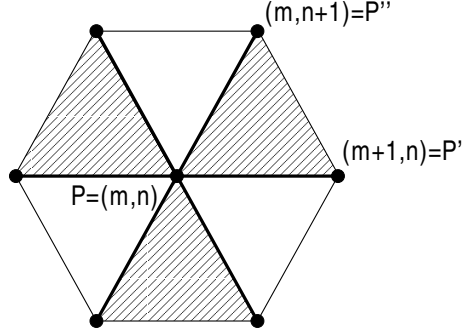


Figure 6:

The coefficients b, c, d correspond to the edges $R = PP', PP'', P'P''$ where $P = (m, n), P' = (m + 1, n), P'' = (m - 1, n + 1)$.

This is a partial case of the triangulated 2-manifold with black-white coloring of triangles—see paragraph 1. We have here a natural isomorphism between the sets of black and white triangles with set of vertices (m, n) . Therefore our operators Q^+, Q and Q'^+, Q' map the space of functions of vertices into itself. Laplace transformations can be applied here infinite number of times. Infinite Chains of Laplace transformations are necessary to define discretization of the 2D Toda lattice. We are going to discuss this problem below.

No gauge transformations are allowed in this case because the transformation $L \rightarrow f^{-1}Lf, \psi \rightarrow f^{-1}\psi$ leads to nonself-adjoint operators, and $L \rightarrow fLf$ destroys potentials W, W' where $L = Q^+Q + W = Q'^+Q' + W'$. For the discretization of 2D Toda lattice we need to consider Laplace transformations associated with all factorizations $L = Q_j^+Q_j + W_j$ corresponding to basic shifts $T_{j,1}, T_{j,2}, j = 0, 1, 2, 3, 4, 5$ where $T_{0,1}, T_{0,2} = T_1, T_2$ and other bases are the rotations of this one by the angle $2j\pi/6$. The bases with numbers j and $j + 3(mod 6)$ are inverse to each other.

Another choice of representative in the gauge class making the Laplace Transformations can be also used for discretization of the 2D Toda Lattice here reducing operator L in the zero level gauge class $L\psi = 0$

$$L \rightarrow fLf, \psi \rightarrow f^{-1}\psi$$

to the form $W = const$ with $const = 0$ or $const = 1$ Here we have

$$L = Q_j^+Q_j + const \rightarrow \tilde{L}_j = Q_jQ_j^+ + const = \tilde{\psi}_j$$

$$\psi \rightarrow Q_j \psi = \tilde{\psi}_j$$

Every time we have to make new factorization and after that perform the "zero level" gauge transformation reducing potential to the constant. The whole collection of Laplace transformations in this case should be optimally organized using relations between them

It is clear that relationships between factorizations and Laplace transformations allow to organize the whole picture in the form of the 3D cubical lattice where our equilateral triangle lattice is imbedded in as a antidiagonal 2D sublattice (i.e. sum of 3 coordinates should be equal to zero). We are going to develop this idea of construction in the next work.

Theorem 3. *The black and white operators $Q^b = Q_j = u_j + v_j T_{j1} + w_j T_{j2}$ and $Q^w = P_j = u'_j + v'_j T_{j1}^{-1} + w'_j T_{j2}^{-1}$ as above form together the SL_2 discrete connection $\{Q^w, Q^b\}$ which does not depend on j .*

Proof. Easy to see that this connection exactly coincides with connection defined by the factorization of real self-adjoint operators in the paragraph 1 written in terms of 3 different pairs of the shift operators.

By the Theorem proved in the paragraph 2 above, this connection is always SL_2 because the product relations between the coefficients of the black and white operators immediately follows from the fact that these 2 different factorizations represent the same operator L as it was pointed out above. So our theorem is proved.

Our Conclusion is that canonical SL_2 connection is associated with real self-adjoint operator on the equilateral triangle lattice. It is analogous to the magnetic part of Schrodinger operators in the Quantum Mechanics. Its behavior under Laplace Transformations and construction of triangle discretization of the 2D Toda Lattice will be discussed later.

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